An Eclectic Approach to the Teaching of Calculus

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Abstract

There are several approaches to the teaching of Calculus, most of which contribute positively to the understanding of this difficult subject. We have developed a path to teach freshman calculus from a multiple perspective, which combines technology, core mathematical ideas and applications to biology, chemistry and physics. The approach presented in this paper stems from a four-year project integrating biology, mathematics, and statistics.

1. Introduction

After teaching introductory calculus on and off for several decades at the college level, I remain skeptical about adopting a one-strategy approach. Interesting and innovative projects have been implemented in the past, for instance, either using computers to teach calculus in a laboratory setting **[1]** or teaching calculus and biology through an integrated approach **[2]**. Unfortunately, to our knowledge none of these projects can be reproduced easily. Moreover, it is difficult to assign students randomly to a treatment and a control group, and there are many variables that a researcher in mathematics education must take care of. The lack of a definitive answer to the question of which is the best method to teach calculus suggests the advisability of combining several approaches, with the objective of reaching a balance between them **[3]**. This paper will try to convey how we address the challenge of balance.

2. Tending a bridge from algebra and geometry

A crucial time for most students is the transition between high school and college. Before discussing limits, derivatives, integrals, and the like, it is advisable to solve several problems that require only algebra and geometry. Here is a short list:

1. Among all rectangles of given fixed perimeter, which one encloses the largest area?
2. Given a chord, build the rectangular pen of maximal area next to a river.
3. Among all right triangles with fixed hypotenuse, which one encloses the largest area?
4. If we wish to achieve maximal volume, what should be the angle at the vertex of a prism-like tent whose vertical walls are isosceles triangles?
5. Assuming that we wish to use the least amount of pipe, where should a pump be located to provide water from a river to two cottages situated on the same shore?

The first two problems lead to quadratic functions while the next two lead to biquadratic functions, thus no calculus techniques are needed. For instance, to solve the third problem we let b denote the length of the hypotenuse and x, y the length of the legs. Since $x^2 + y^2 = b^2$, the area function will be given by $A(x) = \frac{x}{2}\sqrt{b^2 - x^2}$. Consequently $A^2(x) = -\frac{1}{4}x^4 + \frac{b^2}{4}x^2$. Completing squares, we can show that this biquadratic function adopts its maximum at

$$x_{max} = \sqrt{\frac{-b^2/4}{2(-\frac{1}{4})}} = \sqrt{\frac{b^2}{2}} = \frac{1}{\sqrt{2}}b.$$

But $A(x)$ and $A^2(x)$ adopt their maximum at the same point, a fact that is not hard to prove. Then

$$y_{max} = \sqrt{b^2 - (x_{max})^2} = \sqrt{b^2 - b^2/2} = \frac{1}{\sqrt{2}}b.$$

The solution to the problem happens to be the right isosceles triangle! On the other hand, the two-cottages problem can be solved using basic Euclidean geometry only (by a reflection principle). How about solving the isoperimetric problem for isosceles triangles? In other words, among all isosceles triangles of fixed perimeter, which one encloses the largest area? Indeed, if the length of the equal side is y and the third side is $2x$, we will have $2y + 2x = p$. Hence

$$A(x) = x\sqrt{y^2 - x^2} = x\sqrt{(s - x)^2 - x^2} = x\sqrt{s^2 - 2sx}$$

where s is the semiperimeter. Both $A(x)$ and its square attain their maximum at the same point. Thus, we have to find where the function

$$f(x) = x^2(s^2 - 2sx) = -2sx^3 + s^2x^2$$

adopts its maximum. This time we have to deal with a third degree polynomial for which no simple algebraic or geometric approach is available. The Arithmetic-Geometric-Mean inequality could solve the problem for any type of triangles (not necessarily isosceles triangles), but this approach is hardly elementary. The tools of calculus will lead, later in the semester, to an elegant and fast solution, namely $x = s/3$. Therefore $2x = p/3$ and $y = s - \frac{s}{3} = 2s/3$. That is to say, the equilateral triangle is the best option.

The power and limitations of a non-calculus approach can also be seen when analyzing the problem of finding tangents to curves. Suppose we wish to find the equation of the tangent to the parabola $y = x^2$ at (a, a^2). The equation of any secant through (a, a^2) is $y - a^2 = m(x - a)$, thus the points of intersection between the parabola and the secant are the solutions of the equation $x^2 - a^2 = m(x - a)$, that is, $x^2 - mx + (ma - a^2) = 0$.

The tangent being discussed has a particular property, namely, it intersects the parabola at just one point. Algebraically speaking, this means that the discriminant, D, of the above-mentioned quadratic has to be zero:

$$0 = D = m^2 - 4(ma - a^2).$$

Therefore $(m-2a)^2 = 0$, which in turn leads to $m = 2a$. We have reached the conclusion that the equation of the tangent to the parabola at (a, a^2) is $y - a^2 = 2a(x-a)$.

The method under consideration can be applied to any conic **[4]**. For instance, let $f(x) = \frac{1}{x}$. The equation of any secant that passes through $(a, \frac{1}{a})$ is $y - \frac{1}{a} = m(x-a)$. Thus, we have to analyze the roots of

$$\frac{1}{x} - \frac{1}{a} = m(x-a),$$

which is equivalent to $mx^2 + \left(\frac{1}{a} - ma\right)x - 1 = 0$. The discriminant has to be zero, hence $0 = D = (ma - \frac{1}{a})^2 + 4m$. Consequently $0 = m^2a^2 + 2m + 1/a^2$, that is, $(ma + \frac{1}{a})^2 = 0$. Therefore $m = -1/a^2$. We have reached the conclusion that the equation of the tangent to the curve $y = \frac{1}{x}$ at $(a, \frac{1}{a})$ is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x-a).$$

Unfortunately, the elementary algebraic method becomes problematic for non-conic curves such as $y = x^3$. Students will soon learn how the tools of calculus provide a path to the solution of a wide variety of problems linked to tangents. Having previously seen some of the limitations of algebraic and geometric tools, they will surely realize how powerful the calculus machinery is. Thus, students might be better motivated to spend time and effort to master calculus. Every instructor would agree that motivation is one of the key factors that contribute to the success of a calculus course.

3. Core Mathematical Ideas

We start teaching limits by discussing the idea of convergence of a sequence and the main propositions about convergent sequences. All the statements are presented with care, but not much time is devoted to rigorous proofs; after all, this is not a real analysis course. Thereafter, geometric series are analyzed; other types of series will have to wait until Calculus II. It just happens that many interesting examples involve geometric series, thus it makes sense to introduce them rather early (bouncing balls, multiple dosage of drugs, etc.)

It has been our experience that students understand better the idea of the limit of a sequence rather than any other type of limit, with the added advantage that the area under a curve like $y = x^2$, between 0 and $b > 0$ can be calculated before the machinery of integral calculus has been developed. Indeed, it requires calculating $lim_{n\to\infty} A_n$ where

$$A_n = \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}\frac{n}{6}(2n^2+3n+1) = \frac{b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right).$$

Since $\frac{3}{n} \to 0$ and $\frac{1}{n^2} \to 0$, we will have that $A_n \to \frac{b^3}{6} \times 2 = \frac{b^3}{3}$.

The Intermediate Value Theorem and the Extreme Value Theorem are stated but not proven. Similarly, no proof is provided for the most general setting of the chain rule. However, particular cases like $\sqrt{\cos x}$ or $f^2(x)$ are fully justified. The geometric evidence of the Mean Value Theorem (MVT) is overwhelming, hence no formal proof is attempted; many students accept the geometric evidence as a valid justification **[5]**. Rather we spend time on the corollaries of MVT, especially the one that asserts that two functions differ by a constant if their derivatives are equal on an interval. Then the field is open to deal with anti-derivatives and solve simple differential equations, well before the definite integral is introduced through sequences of Riemann sums.

As expected, a high point of first-year calculus is the discussion of the Fundamental Theorem of Calculus (FTC). We only prove a particular case of it, namely the case when the function is continuous, positive, and monotonic **[6]**, but state and use FTC in its full generality. The path is then ready to define the natural logarithm and its inverse, which happens to be the exponential function. Actually, we are revisiting both functions because they were introduced quite early accepting the fact that the derivative at the origin of e^x is 1.

We judge that our students will learn to write short proofs of non-evident questions, in a style that is not too formal **[7].** This is realistic because many of them have had some contact with calculus at the high school level. But these are not proofs that involve epsilon-delta type arguments. There seems to be a consensus in the sense that it is unrealistic to introduce this type of arguments in an introductory calculus course, not only in the U.S. but in France too **[8]**. In North American colleges and universities, most mathematics majors have to wait until their junior year before they take a real analysis course. Nonetheless, it has been our experience that several top first-year college students can profit from enrichment notes, wherein they are exposed to a formal approach to calculus that goes beyond the established curriculum.

4. Graphing Calculators

Modeling and the scientific method are ideas that should be introduced as early as possible in a first-year calculus course. There is the need to teach the concepts of correlation and linear regression, two topics of basic statistics that can be discussed right after the section on maxima and minima. We might start analyzing a particular case of the least squares regression line, namely when it is assumed that the “line of best fit” passes through the origin, before dealing with the more general case. The latter involves a function of two variables, a topic that goes beyond the comprehension of most first-year students. Thus, it is a wise idea to discuss the regression through the origin case in detail and thereafter display, without proof, the more general case of the least squares regression line. Then the gate to technology is open and students are ready to use the LinReg option from their graphing calculators or any available statistical software. Luckily, many students have already encountered in High School, one way or another, the ideas behind linear regression; or they take an introductory statistics course before or concurrently with calculus along the pattern set by contemporary textbooks **[9]**.

Data on many phenomena is readily available from the literature, especially biological data. Examples include growth of fish, length of wings and weight of birds, first and second order chemical kinetics, Newton's law of cooling, exponential growth, logistic growth, Torricelli's law, etc. A model is put forward, usually a simple differential equation is involved (either linear of the first or second order, or of the separable variables type) and it is our task to obtain an expression from it that can be used to compare with data and estimate the parameters of the model. Not only students learn how scientists accept or reject models, but they themselves can collect data in simple experiments linked either to the law of cooling or Torricelli's law.

As an illustration, let us consider the growth of a species of fish . These animals tend to grow while they are alive, but the rate is not constant. If L denotes the final length of a mature individual, we might put forward the hypothesis that the rate of growth is proportional to the difference between L and $x(t)$ (the length at any time t). That is to say,

$$x'(t) = k\big(L - x(t)\big), \tag{1}$$

where k is a parameter. Dividing by $L - x(t)$ and integrating we get

$$\int \frac{-x\prime(t)}{L-x(t)}\,dt = \int -k dt.$$

Therefore

$$\ln\big(L - x(t)\big) = -kt + C \tag{2}$$

for a certain constant C. A linear regression analysis will provide the least squares regression line, whose slope is then used as an estimate of k. Let us see whether data **[10]** (Table 1) related to *Coregomus clupeaformis* (North American lake whitefish) is in agreement with (2).

t (years)	x(t) (cm)
1	15
2	23
4	42
6	53
8	58
10	64
L	71

Table 1

We need to build a table with time in the first column and $\ln(71 - x(t))$ in the second column (Table 2).

t	y=ln(71-x(t))
1	4.0254
2	3.8712
4	3.3673
6	2.8904
8	2.5649
10	1.9459

Table 2

A linear regression analysis , using the LinReg option of a graphing calculator, provides the line of best fit:

$$\hat{y} = -0.228138t + 4.289559.$$

Since the correlation coefficient is $r_{ty} = -0.997212$, a number very close to -1, the model $x'(t) = k(L - x(t))$ seems adequate and we proceed to estimate k as the number -0.228138. With the estimation 4.289559 for C we finally get

$$x(t) = 71 - e^{4.289559}e^{-0.228138\,t} = 71 - 72.9342e^{-0.228138\,t}.$$

Besides providing fast and accurate answers to the problem of correlation and linear regression, graphing calculators are ideally suited to compute indefinite integrals like

$$\int e^{-0.3t}\sin(1.7t)\,dt \qquad \text{or} \qquad \int \frac{dx}{a - \frac{x}{x+b}}\,.$$

Either one can be solved by hand; however, doing so requires time and patience: the first integral requires applying integration by parts twice, while the second integral involves a process of long division. The latter integral appears in the discussion of the phenomenon of facilitated diffusion, a type of transport across cell membranes. The basic model is

$$\frac{dS_i}{dt} = \frac{S_e V_{max}}{S_e + K_m} - \frac{S_i V_{max}}{S_i + V_{max}}, \qquad (3)$$

where S_e denotes the concentration of substance outside the cell and S_i denotes the concentration of substance inside the cell (K_m and V_{max} are parameters). We have to obtain mathematical consequences of (3) and compare them with data. With this idea in mind, let us start by assuming that the concentration of substance on the outside is much bigger than in the inside of the cell; thus, S_e can be considered constant. Then

$$\int \frac{dS_i}{a - \dfrac{S_i}{S_i + b}} = \int V_{max}\, dt,$$

where $a = \frac{S_e}{S_e+K_m}$ and $b = K_m$. Using a graphing calculator or a computer we get

$$\int \frac{dx}{a-\frac{x}{x+b}} = \frac{x}{a-1} - \frac{b}{(a-1)^2}\ln|(a-1)x + ab|.$$

Thus

$$\frac{S_i}{\frac{S_e}{K_m+S_e}-1} - \frac{K_m}{\left(\frac{S_e}{K_m+S_e}-1\right)^2}\ln\left[\left(\frac{S_e}{K_m+S_e}-1\right)S_i + \frac{K_m S_e}{K_m+S_e}\right] = V_{max}t + L,$$

for a certain constant L. A little bit of arithmetic leads to

$$S_i(K_m + S_e) + (K_m + S_e)^2 \ln\left[\frac{K_m}{K_m + S_e}(S_e - S_i)\right] = -K_m V_{max} t + H,$$

where $H = -LK_m$. If we assume that $S_i = 0$ at $t = 0$, right away we get

$$H = (K_m + S_e)^2 ln\frac{K_m S_e}{K_m+S_e}.$$

Replacing this value in the preceding equation, we finally reach the equality

$$S_i(K_m+S_e) + (K_m + S_e)^2 ln\frac{S_e-S_i}{S_e} = -K_m V_{max} t. \qquad (4)$$

The challenge is to estimate the values of K_m and V_{max} from (4). For this purpose, the half-equilibrium time method can be used **[11]**.

Of course, students need to know how to solve simple integrals using change of variables or integration by parts, but they should not be overwhelmed with subtle substitutions that are time-consuming.

A less common use of a graphing calculator has to do with programming. Students can learn how to enhance the usefulness of their electronic devices by writing simple programs, with the help of their teachers, to implement Newton's method or approximate a definite integral through a weighted average of the trapezoid and the midpoint approximation, which happens to be Simpson's method with $2n$ steps. The programs will work only if the students thoroughly understand the underlying mathematical concepts. For instance, if students learn that Newton's approximation method involves the recursive process $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, the path to the following program (using a TI-89 or TI-92) does not demand an inordinate amount of work:

```
: newt( )
: Prgm
: Input "xo", x
: Input "n", n
: For  i, 1, n
: x − y1(x)/y2(x) → x
: Disp x
: Endfor
: Endprgm
```

They will also learn, as a byproduct, what the For command does and the need to upload the given function as $y1(x)$ and its derivative as $y2(x)$. On the other hand, the following program allows students to approximate a definite integral between two points *a* and *b*:

```
TMA()
: Prgm
: Input "a", a
: Input "b", b
: Input "n", n
: a + (i − 1) * (b − a)/n → p
: a + i * (b − a)/n → q
: a + (i − 1/2) * (b − a)/n → u
: (b−a)/n * ∑(y1(p), i, 1, n) → l
: (b−a)/n * ∑(y1(q), i, 1, n) → r
: (b−a)/n * ∑(y1(u), i, 1, n) → m
: Disp  l
: Disp  r
: Disp  (l + r)/2
: Disp  m
: Disp (2m + ((l+r)/2))/3
: Endprgm
```

It is to be noted that under the $y1$ option of the graphing calculator we write the function that we are interested in and once we write TMA() under the Home option , and the program is run, five numbers will appear on the screen: the first corresponds to the left approximation, the next gives us the right approximation, the third is the trapezoidal approximation, while the fourth is the midpoint approximation. The last one, and the closest to the "true value" of the integral, is the weighted average between the midpoint and the trapezoidal approximation. Research confirms that there is a conceptual gain when one engages in writing programs **[12]**.

A more "standard" use of graphing calculators involves the all-important task of exploration. For instance, if we are asked to find the points of inflection of $f(x) = \frac{1}{1+e^{-x}}$, we might better make a graph before calculating the first and second derivatives. Or, if asked to determine whether the improper integral $\int_0^\infty \frac{1}{e^x+e^{-x}}dx$ is convergent, we might start by using technology to evaluate $\int_0^{10} \frac{1}{e^x+e^{-x}}dx$ and $\int_0^{15} \frac{1}{e^x+e^{-x}}dx$. It seems that the integral is convergent: the first definite integral is approximately 0.78535276 while the second is approximately 0.78539786. Thereafter we can compare the given improper integral with $\int_0^\infty e^{-x}\,dx$ and reach the conclusion that it is indeed convergent. Moreover,

$$\int \frac{1}{e^x + e^{-x}}dx = \int \frac{e^x}{e^{2x}+1}dx = \int \frac{du}{u^2+1} = \arctan u + C = arctan\,(e^x) + C.$$

Therefore,

$$\int_0^b \frac{1}{e^x+e^{-x}}dx = [\arctan(\mathrm{e}^{\mathrm{b}}) - \arctan(1) \to \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \text{as } b \to \infty.$$

Thus $\int_0^\infty \frac{1}{e^x+e^{-x}}dx = \frac{\pi}{4}$.

Technology certainly need not hamper student's learning of calculus. On the contrary, there is evidence that it might even help them provided that they learn some basic "paper and pencil" techniques too **[13]**. Of course, computers could be used instead of graphing calculators if a sufficient number of machines were available. Computer laboratories in many departments of mathematics and statistics are intensively used in statistics courses and upper division mathematics courses, but sometimes do not have the infrastructure to serve hundreds of first-year calculus students.

5. Applications to the Natural Sciences

My vision about the role of applications in an introductory calculus course has been heavily influenced by the Symbiosis Project (2006-2010) at East Tennessee State University, a systematic attempt to integrate biology, mathematics, and statistics during the first three semesters of studies. Two cohorts of biology students followed the experimental curricula: the first semester was mostly statistics intertwined with biology, the second semester dealt with calculus linked to biology, while the third semester revolved around bioinformatics, including microarray data analysis, and other mathematical and statistical topics **[14]**. For a wider audience of biology, chemistry, and physics students, we developed a freshman calculus course heavily oriented toward applications to the natural sciences, without compromising the conceptual aspects of mathematics. Besides the usual examples from physics, we include many examples from biology and chemistry in the belief that they are models of real phenomena that illustrate quite well the underlying ideas from one-variable calculus. For instance, we discuss extensively several quantitative aspects of enzyme kinetics. Let us recall Michaelis-Menten equation, which asserts that, at any instant t during the steady-state,

$$v = \frac{V_{max}S(t)}{S(t)+K_m},$$

where $v = P'(t) = -S'(t)$ ($P(t)$ denotes the concentration of product while $S(t)$ denotes the concentration of substrate; K_m and V_{max} are parameters that involve the kinetic constants of the chemical reaction $S + E \rightarrow C$ (a reversible process) and $C \rightarrow E + P$ (irreversible process), with E representing the enzyme and C the intermediate compound. Thus

$$-S'(t) = \frac{V_{max}S(t)}{S(t)+K_m}.$$

Multiplying by $\frac{S(t)+K_m}{S(t)}$ we arrive at $-S'(t) - K_m \frac{1}{S(t)} S'(t) = V_{max}$. Then, integrating with respect to time between 0 and t (considering 0 as the instant when the steady state starts), we get

$$-\int_0^t S'(u)du - K_m \int_0^t \frac{S'(u)}{S(u)} du = \int_0^t V_{max} du.$$

Consequently

$$(-\big(S(t) - S(0)\big) - K_m ln \frac{S(t)}{S(0)} = V_{max}t,$$

which in turn leads to

$$\frac{1}{t} ln \frac{S(0)}{S(t)} = -\frac{1}{K_m} \frac{S(0) - S(t)}{t} + \frac{V_{max}}{K_m}.$$

This is a remarkable expression because it does not involve rates but only experimental values of $S(t)$ during the course of an enzymatic reaction. It predicts the appearance of a line if we have the quotient $\frac{S(0)-S(t)}{t}$ on the horizontal axis and $\frac{1}{t} ln \frac{S(0)}{S(t)}$ on the vertical axis; it is a line with slope $-1/K_m$ and vertical intersection V_{max}/K_m . Thereafter we build a table with $\frac{S(0)-S(t)}{t}$ in the first column and $\frac{1}{t} ln \frac{S(0)}{S(t)}$ in the second column. A linear regression analysis will provide us with an estimation to the slope $-1/K_m$ and the vertical intersection V_{max}/K_m. Finally, a simple arithmetical procedure will lead to the corresponding values of K_m and V_{max} .

Students usually respond positively to the introduction of many applications to the natural sciences. However, we should always keep in mind what L.A. Steen **[15]** wrote about the advantages and perils of integrating the teaching of mathematics with science:

" It is easy to imagine several possibilities for how the proposed change might come about. [One of them is to employ] mathematical methods thoroughly in science, and scientific methods thoroughly in mathematics, coordinating both subjects sufficiently to make this feasible. This is, I submit, an ideal situation. Each discipline, science and mathematics, would accrue benefits from an infusion of methods of the other, but neither would loose its identity or distinguishing features in an artificial effort at union. There are, after all, important differences between science and

mathematics, both philosophical, methodological, and historical. These should not be lost in a misguided effort at homogenization..

6. How well do students perform in Calculus?

Students enrolled in the Symbiosis Project took the Gateway Exam on Calculus given to all regular sections and did as well as any other group of students **[16]**. Once finished with the Symbiosis Project, the Department of Mathematics and Statistics at East Tennessee State University has been offering a special freshman calculus section, recommended for biology, chemistry and physics majors. Therein we have used a new textbook **[17]** , which follows the approach mentioned in the previous section. They all passed the Gateway exam and, at the end of semester evaluation of the instructor (a four-point semantic differential Likert scale), the mean score was 31 (8 questions, from "strongly agree" to "strongly disagree", with 32 as the highest achievable mean score of satisfaction). It should be mentioned that students were not randomly assigned to the special section and that the evaluation may reflect, in some way, the degree of rapport between students and instructor rather than any significant cognitive gain as compared to the regular calculus sections.

7. Conclusions

It seems highly advisable to address the scope and limitations of algebra and geometry before a formal study of calculus. Thereafter, a gradual introduction of the main mathematical ideas of differential and integral calculus, coupled with a rational use of technology, sets the path to follow. The proofs developed in class should be limited in number and not necessarily the most general available, but every proposition needs to be clearly stated. Not discussing subtle traditional topics, which can be handled by technology, saves precious time to analyze relevant applications to the natural sciences.

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